

Elementary gates of ternary quantum logic circuit

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The elementary gates are basic building blocks of quantum logic circuit. They should be simple, efficient, and easy to implement. In this article, we propose the ternary controlled X (TCX) gate or the ternary controlled Z (TCZ) gate as the two-qutrit elementary gate, which is universal when assisted by arbitrary one-qutrit gates. Based on Cartan decomposition, we give the one-qutrit elementary gates. Also, we discuss the physical implementation of these elementary gates and show that they are feasible with current technology. Then we investigate the synthesis of some important ternary gates, such as the ternary SWAP gate, ternary Toffoli gates and Muthukrishnan-Stroud gates. Finally we extend these elementary gates to a more general case for qudit systems. This work provides a unified description for the synthesis of both binary and multi-valued quantum circuits.

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I. INTRODUCTION

Quantum computer has attracted a great deal of attention due to its potentialities to solve classical NP problems in polynomial time. In quantum computing, the algorithms are commonly described by the quantum circuit model [1]. In 1995, Barenco *et al.* showed that any binary quantum logic circuit can be decomposed into a sequence of one-qubit gates and CNOT gates [2]. The process of constructing quantum circuits by these elementary gates is called synthesis by many authors. The complexity of quantum circuit can be measured in terms of the number of elementary gates required. Achieving gate arrays of less complexity is crucial as it reduces not only the resource but also the errors.

Most approaches to quantum computing use two-level quantum systems (qubits). Recent studies have indicated that there are some advantages to expand quantum computer from qubits to multi-level system (qudits). Three level quantum systems, so called qutrits, are the simplest multi-valued systems. There have been many proposals to use multi-level quantum systems to implement the quantum computation and other quantum information processes [3–8]. In experiment, there have been reports on their applications in simplifying quantum gates [9], simulating physical system with spin greater than 1/2 [10]. Multi-level quantum systems have been realized in many ways in the field of optics [11–13]. In solid-state devices, the experimental demonstrations of full quantum state tomography of the qutrit have been reported recently [14, 15]. But multi-valued quantum logic synthesis is still a new and immature research area. The crucial issue which gates is chosen as the elementary gate set of multi-valued quantum circuit is not well solved. The elementary gates are the basic blocks for constructing quan-

tum logic circuit, and they should be universal, simple, effective and easy to implement.

A number of works have been done on multi-valued logic synthesis by some authors. In 2000, Muthukrishnan and Stroud investigated the synthesis of multi-valued quantum circuit [4] and showed that two two-qudit gates, which are called Muthukrishnan-Stroud gates now, together with the one-qudit gates, are universal for quantum computing. In 2002, Perkowski, Al-Rabadi, and Kerntopf proposed a set of generalized ternary gates (GTG gate) [16] based on a ternary condition gate and ternary shift gates [17]. During 2005 to 2006, based on cosine-sine decomposition of matrix [18], the synthesis of ternary and more general multi-valued quantum logic circuits was investigated by Khan and Perkowski in Refs.[19] and [20], respectively. The multi-valued quantum circuit can be synthesized in terms of quantum multiplexers and uniformly controlled rotations. But these components themselves have a complicated structure and their synthesis needs study further.

On the other hand, two Brylinskies [21] proved that any two-qudit gate that creates entanglement without ancillas can act as a universal gate for quantum computation, when assisted by arbitrary one-qudit gates. That is to say, “almost every” two-qudit gate is universal when assisted by one-qudit gates. But not all these gates are suitable to be chosen as the two-qudit elementary gate of the quantum multi-valued circuit. Just as the binary quantum circuit, we usually choose CNOT gate or controlled-Z gate as the two-qubit elementary gate although “almost every” two-qubit gate is universal. Two Brylinskies’ proof relies on a long argument using advanced mathematics. No any specific gate is proposed as the two-qudit elementary gate. Alber investigated the purification of bipartite high dimension quantum states with hermitian generalized XOR gate (GXOR gate) *et al.* [22]. As a universal bipartite gate for ternary quantum computing, a protocol of the physical implementation of the GXOR gate on ions in a trap was presented by Klimov

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et al. in [23]. Wang *et al.* discussed the entanglement power of operators in qudit systems. They proposed to choose the SUM gate, which is called Feynman gate in computer community, as the elementary bipartite gate for qudit quantum computing [24]. But little work has been done based on these gates for the specific synthesis of multi-valued quantum circuit.

In this article, we focus on the investigation of elementary gates of ternary quantum logic circuits. We propose the ternary controlled X (TCX) gate or the ternary controlled Z (TCZ) gate as the two-qutrit elementary gate. Based on the Cartan decomposition [25], the one-qutrit elementary gates are also given. The elementary gates proposed here are essentially binary and which can be implemented with current technology. Also, we expand these elementary gates to a more general case of qudit systems. So many results in binary quantum logic circuits can be generalized to a multi-valued case. They can be used as a unified measure of complexity for various quantum logic circuits.

This article is organized as follows. In Sec. II, we investigate two-qutrit elementary gate for ternary quantum logic circuits and propose the TCX gate or the TCZ gate as the elementary gate. In Sec. III, based on Cartan decomposition we discuss the set of one-qutrit elementary gates and the synthesis of generic one-qutrit gates. The physical implementations of these elementary gates are studied in Sec. IV. And the synthesis of some important ternary quantum gates, such as the ternary SWAP gate, ternary Toffoli gates and Muthukrishnan-Stroud gates, is given in Sec. V. In Sec. VI, we extend our study to a general case for qudit systems. Finally a brief conclusion and future work is given in Sec. VII.

II. TWO-QUTRIT ELEMENTARY GATES

In one qutrit case, there are three X quantum gates given by the matrices

$$\begin{aligned} X^{(01)} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ X^{(02)} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ X^{(12)} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (1)$$

Similarly, for every single qubit gate A, we can simply extend it to a set of ternary gate, $A^{(ij)}$. The ternary extension of Hadamard gate can be expressed as

$$H^{(01)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

$$\begin{aligned} H^{(02)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}, \\ H^{(12)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \end{aligned} \quad (2)$$

The extension of Z gate is slightly different, we denote them as

$$\begin{aligned} Z^{[0]} &= \text{diag}\{-1, I_2\}, Z^{[1]} = \text{diag}\{1, -1, 1\}, \\ Z^{[2]} &= \text{diag}\{I_2, -1\}. \end{aligned} \quad (3)$$

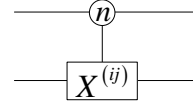


FIG. 1: Ternary controlled-X gate.

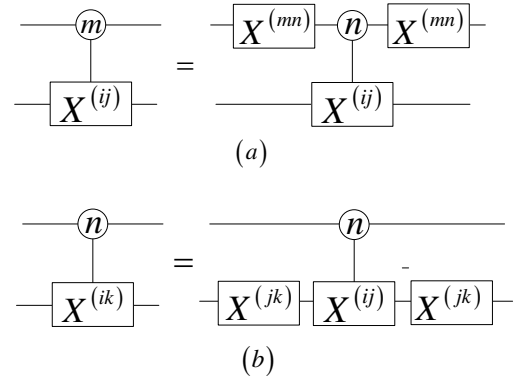


FIG. 2: Transformation among different TCX gates. (a) Transformation of control mode. (b) Transformation of target operations.

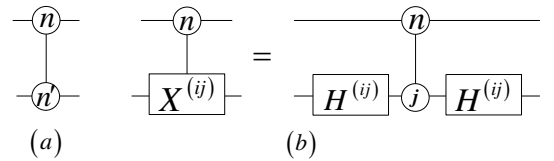


FIG. 3: TCZ gates and transformation between TCX and TCZ gates.

The TCX gate is a two-qutrit gate which is defined that the gate implements the $X^{(ij)}$ operation on the target qutrit iff the control qutrit is in the states $|n\rangle$, ($n \in \{0, 1, 2\}$). The circuit representation for the TCX gate is shown in Fig. 1 in which the line with a circle

represents the control qutrit, while that with a square represents the target qutrit. There are nine different forms for the TCX gate and they can be easily transferred one another as shown in Fig. 2. The TCZ gate is defined that the gate implements the $Z^{[n']}$ operation on the target qutrit iff the control qutrit is in the states $|n\rangle$. Similar to the binary controlled Z gate, the control qutrit and target qutrit of TCZ gate are exchangeable. It has nine different forms, which also can be transferred one another by using ternary X gates. The symbol of TCZ gate and its transformation relation with TCX gate are shown in Fig. 3.

The ternary shift gates proposed in Ref. [17] are basic one-qutrit gates. Operations, symbols and the relations with X operations are listed in Fig. 4. Likewise, the two-qutrit controlled shift gates can be defined. Two-qutrit Feynman gate which is called SUM gate in [24] is shown in Fig. 5. Here A is the controlling input and B is the controlled input. The output in control qutrit equals to the input A , and the output in target qutrit is the sum of A and B modulo 3. The synthesis of the ternary Feynman gate base on the TCX gates is shown in Fig. 6. GXOR gate is similar to the Feynman gate, and the difference is that the output is the difference of A and B modulo 3. Its synthesis is shown in Fig. 7. The GTG gates are the combinations of the controlled shift gates and they are mainly used to investigate the synthesis of permutation quantum gates by some groups [16, 17]. The permutation gate is a gate which unitary matrices have only one 1 in every column and the remaining elements 0.

Gate Name	Symbol & Operation	Relation with X operation
Buffer	$x \rightarrow \triangle \rightarrow x$	I
Single-shift	$x \rightarrow \triangle' \rightarrow x' = x + 1$	$X^{(01)} X^{(12)}$
Dual-shift	$x \rightarrow \triangle'' \rightarrow x'' = x + 2$	$X^{(12)} X^{(01)}$
Self-shift	$x \rightarrow \triangle''' \rightarrow x''' = 2x$	$X^{(12)}$
Self-single-shift	$x \rightarrow \triangle^\# \rightarrow x^\# = 2x + 1$	$X^{(01)}$
Self-dual-shift	$x \rightarrow \triangle^\wedge \rightarrow x^\wedge = 2x + 2$	$X^{(02)}$

FIG. 4: Ternary shift gates.

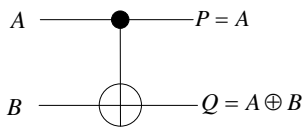


FIG. 5: Ternary Feynman gate.

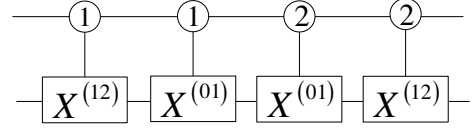


FIG. 6: Synthesis of ternary Feynman gate.

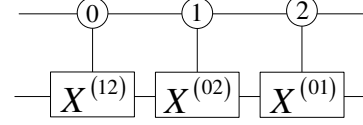


FIG. 7: Synthesis of GXOR gate.

The X operation is more elementary than the shift operation. From the point of view of group theory, the six shift operations constitutes a permutation group S_3 , and the $X^{(01)}$ and $X^{(12)}$ operations are the generators of the group. The TCX gate is an elementary counterpart of the binary CNOT gate. It is simple and easy to implement. While the Feynman gate, GXOR gate and GTG gates have complex structures themselves. So we propose the TCX gate as the two-qutrit elementary gate for qutrit-based quantum computation. Of course, the choice is not unique, and the TCZ gate also can be chosen as the two-qutrit elementary gate.

III. ONE-QUTRIT ELEMENTARY GATES

The complexity of binary quantum logic gate is usually measured by the numbers of CNOT gate and one-qubit R_y , R_z gates. We call the R_y , R_z gates as one-qubit elementary gates. Suppose M is the matrix of a one-qutrit gate. Using the Cartan decomposition of Lie group, it can be expressed [26] as

$$M = e^{i\alpha} R_y^{(01)}(\beta) R_y^{(02)}(\gamma) R_y^{(01)}(\delta) R_z^{(01)}(\theta) R_z^{(02)}(\varphi) R_y^{(01)}(\beta') R_y^{(02)}(\gamma') R_y^{(01)}(\delta'), \quad (4)$$

where α, β, γ , etc. are all real numbers. Here $R_\alpha^{(jk)}(\theta) = \exp(-i\theta\sigma_\alpha^{(jk)}/2)$ for $j < k$ and $\alpha \in \{x, y, z\}$. $\sigma_x^{(jk)} = |j\rangle\langle k| + |k\rangle\langle j|$, $\sigma_y^{(jk)} = -i|j\rangle\langle k| + i|k\rangle\langle j|$, and $\sigma_z^{(jk)} = |j\rangle\langle j| - |k\rangle\langle k|$. The four basic one-qutrit gates, $R_y^{(01)}$, $R_z^{(01)}$, $R_y^{(02)}$, $R_z^{(02)}$ constitutes a set of one-qutrit elementary gates.

The Cartan decomposition of one-qutrit gates is not unique, so the choice of one-qutrit elementary gates is not unique too. From one kind of Cartan decomposition presented in Appendix A, we get another product expression of a single qutrit gate that

$$M = e^{i\varphi} M_1^{(jk)} M^{(j'k')} M_2^{(jk)}. \quad (5)$$

Here $M^{(jk)}$ is a special unitary transformation in 2-dimensional subspace \mathcal{H}_{jk} , and it can be factored further by the Euler decomposition. The Euler decomposition usually has two modes: ZYZ decomposition and XYX decomposition. So the set of one-qutrit elementary gates has two pairs of basic gates in subspaces \mathcal{H}_{jk} and $\mathcal{H}_{j'k'}$ respectively. We can take $R_y^{(jk)}$, $R_z^{(jk)}$, $R_y^{(j'k')}$, $R_z^{(j'k')}$ or $R_x^{(jk)}$, $R_y^{(jk)}$, $R_x^{(j'k')}$, $R_y^{(j'k')}$ as one-qutrit elementary gates. The set of one-qutrit elementary gate given in Ref. [26] is one of them. The synthesis of generic one-qutrit gates is given by Eq. (4) or Eq. (5).

IV. PHYSICAL IMPLEMENTATION OF TERNARY ELEMENTARY GATES

As we know, there are not many proposals on the physical implementation of ternary gates. In Ref. [4], a scheme for the implementation of Muthukrishnan-Stroud gates based on the linear ion trap is given. Also based on the ion trap, a scheme for the GXOR gate is given by Klimov *et al.* in Ref. [23]. But these schemes and the gates themselves are rather complicated, and no experimental investigations on them have been reported yet. However, in the last decade, there has been tremendous progress in the experimental development of binary quantum computing, and the problem of constructing a CNOT gate has been addressed from various perspectives and for different physical systems [27–39]. The elementary gates proposed here can be implemented by existing technique.

Assume we have a V-type three-level quantum system shown in Fig. 8, which constitutes a qutrit and the two levels of the system $|0\rangle$ and $|1\rangle$ forms a qubit. Two laser beams Ω_1 and Ω_2 are applied to the ion to manipulate $|0\rangle \leftrightarrow |1\rangle$ and $|0\rangle \leftrightarrow |2\rangle$ transition, respectively. If a two-qubit CNOT gate is realized in such systems, one of TCX gate is naturally obtained, and the eight other form TCX gates can be obtained by the transformation shown in Fig. 2. The single qutrit gates are implemented by Rabi oscillations between the qutrit levels. Applying the laser pulses in Ω_1 and Ω_2 and choosing suitable phases, this allows us to perform $R_x^{(01)}$, $R_y^{(01)}$ and $R_x^{(02)}$, $R_y^{(02)}$ gates respectively [1, 40]. So a set of one-qutrit elementary gates is obtained, and any one-qutrit gate can be implemented according to Eq. (4) or Eq. (A8). There are other two types of quantum system, Λ -type and cascade type. We can use $R_x^{(01)}$, $R_y^{(01)}$, $R_x^{(12)}$, $R_y^{(12)}$ or $R_y^{(02)}$, $R_z^{(02)}$, $R_y^{(12)}$, $R_z^{(12)}$ as one-qutrit elementary gates to meet the requirement of manipulating quantum states in these types of quantum system.

It is not too difficult to find such a quantum system. Early in 2003, the Innsbruck group implemented the complete Cirac-Zoller protocol [27] of CONT gate with two calcium ions (Ca^+) in a trap [29]. The energy level scheme of Ca^+ is given in Ref. [31]. The original qubit information is encoded in ground state $S_{1/2}$ and metastable

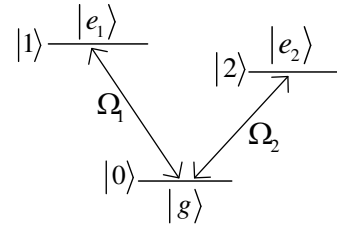


FIG. 8: V-type three level quantum system.

$D_{5/2}$ state. The $D_{5/2}$ state has a lifetime $\tau \simeq 1.16$ s. There is another metastable $D_{3/2}$ state in Ca^+ . Its lifetime, which is measured recently by Kreuter *et al.* [41], is about the same as that of the $D_{5/2}$ state. The three levels of Ca^+ , one ground state and two metastable states, may constitute a qutrit candidate. The CNOT gate was implemented by Schmidt-Kaler *et al.* [29] forms naturally a TCX gate. Two laser pulses are used to manipulate the $S_{1/2} \leftrightarrow D_{5/2}$ quadruple transition near 729 nm and the $S_{1/2} \leftrightarrow D_{3/2}$ transition near 732 nm, respectively. Rabi oscillations between these levels can implement the one-qutrit elementary gates $R_x^{(01)}$, $R_y^{(01)}$ and $R_x^{(02)}$, $R_y^{(02)}$.

The superconducting quantum information processing devices are typically operated as qubit by restricting it to the two lowest energy eigenstates. By relaxing this restriction, we can operate it as a qutrit or qudit. As mentioned in introduction, the experimental demonstrations of the tomography of a transmon-type superconducting qutrit and a superconducting phase qutrit have been reported in Ref. [14] and Ref. [15], respectively. It means that to prepare arbitrary one-qutrit state and read out with high-fidelity on these systems has been implemented. So the one-qutrit gates can be implemented on the systems by the method described here. Construction of a robust CNOT gate on superconducting qubits has been extensively investigated both in theory and experiment [33–38]. So the condition to implement elementary gates of ternary quantum logic circuit has come to maturity on these superconducting qutrits.

V. SYNTHESIS OF SOME IMPORTANT TERNARY QUANTUM GATES

Since the X operations given in Eq. (1) and the four one-qutrit elementary gates all only act on two levels in a qutrit, many results in binary quantum logic circuit can be generalized to ternary quantum logic circuits. The synthesis of binary quantum circuit has been extensively investigated by many groups [1, 2, 42–47], and it is rather mature now. The ternary SWAP gate interchanges the states of two qutrits acted by the gate. It is a generalization of binary SWAP gate and can be decomposed into three binary SWAP gates, i.e.,

$$W = W^{(01)} \cdot W^{(02)} \cdot W^{(12)}. \quad (6)$$

Here the $W^{(ij)}$ can be called as a conditional SWAP gate, and each of them is synthesized by three TCX gates. So the ternary SWAP gate is synthesized by nine TCX gates, as shown in Fig. 9. Likewise, the ternary root SWAP gate can be decomposed into three binary root SWAP gates.

The ternary Toffoli gate has many forms. Here we define an elementary ternary Toffoli gate that two control

qutrits are unaffected by the action of the gate, and the target qutrit is acted by the $X^{(ij)}$ operation iff the two control qutrits are in the states $|n\rangle, |n'\rangle$ respectively. By the result of quantum synthesis for the binary Toffoli gate [1], the synthesis of the elementary ternary Toffoli gate can be obtained and is illustrated in Fig. 10. It needs six TCX gates and ten single qutrit gates, which are the simple extension of single qubit gates H, T, T^\dagger and S .

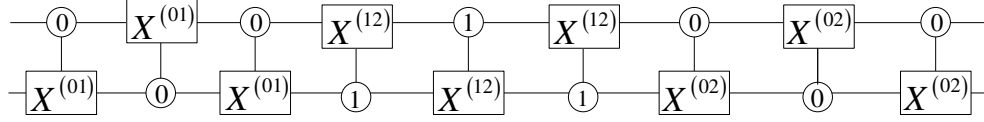


FIG. 9: Synthesis of ternary SWAP gate.

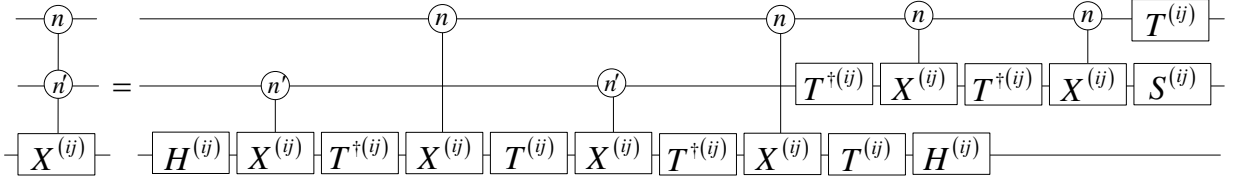


FIG. 10: Synthesis of ternary elementary Toffoli gate.

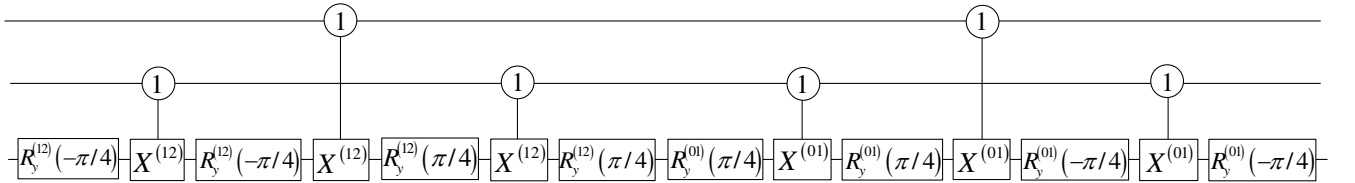


FIG. 11: Synthesis of a typical ternary Toffoli gate.

A typical ternary Toffoli gate is that the two control qutrits remain no change, and the output of target qutrit is $C \oplus 1$ iff both two qutrits are in the state $|1\rangle$, where C is the input of the target qutrit. Yang *et al.* defined a generalized ternary Toffoli gate for multiple qutrit systems in Ref. [48], and it is just this kind of ternary Toffoli gate in the three qutrits case. We give its synthesis shown in Fig. 11, which needs six TCX gates and eight one-qutrit elementary gates.

The two Muthukrishnan-Stroud gates are denoted by $\Gamma_2(\mathbb{Z})$ and $\Gamma_2(\Phi)$ respectively. They are two-qutrit controlled gates in which if the control qutrit is set to $|2\rangle$, then the operation \mathbb{Z} or Φ is applied to the target qutrit respectively. \mathbb{Z} is a family of one-qutrit gates which trans-

fer a definite single qutrit state to the state $|2\rangle$, that is,

$$\mathbb{Z}(c_0, c_1, c_2) : c_0|0\rangle + c_1|1\rangle + c_2|2\rangle \mapsto |2\rangle. \quad (7)$$

It does not determine the transform uniquely. Assume $c_0 = \cos \theta_1 e^{i\varphi_0}$, $c_1 = \sin \theta_1 \cos \theta_2 e^{i\varphi_1}$, and $c_2 = \sin \theta_1 \sin \theta_2$, one of expression of the operation can be written as

$$\mathbb{Z} = PQ = \begin{pmatrix} \sin \theta_1 & 0 & -\cos \theta_1 e^{i\varphi_0} \\ 0 & 1 & 0 \\ \cos \theta_1 e^{-i\varphi_0} & 0 & \sin \theta_1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin \theta_2 & -\cos \theta_2 e^{i\varphi_1} \\ 0 & \cos \theta_2 e^{-i\varphi_1} & \sin \theta_2 \end{pmatrix}. \quad (8)$$

The synthesis of $\Gamma_2(\mathbb{Z})$ based on TCZ gate is shown in Fig. 12. Here

$$\begin{aligned} V_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{\pi}{4} - \frac{\theta_2}{2}) & -\sin(\frac{\pi}{4} - \frac{\theta_2}{2})e^{i\varphi_1} \\ 0 & \sin(\frac{\pi}{4} - \frac{\theta_2}{2})e^{-i\varphi_1} & \cos(\frac{\pi}{4} - \frac{\theta_2}{2}) \end{pmatrix} \\ &= R_z^{(12)}(-\varphi_1)R_y^{(12)}(\frac{\pi}{2} - \theta_2)R_z^{(12)}(\varphi_1), \end{aligned} \quad (9)$$

$$\begin{aligned} V_2 &= \begin{pmatrix} \cos(\frac{\pi}{4} - \frac{\theta_1}{2}) & 0 & -\sin(\frac{\pi}{4} - \frac{\theta_1}{2})e^{i\varphi_0} \\ 0 & 1 & 0 \\ \sin(\frac{\pi}{4} - \frac{\theta_1}{2})e^{-i\varphi_0} & 0 & \cos(\frac{\pi}{4} - \frac{\theta_1}{2}) \end{pmatrix} \\ &= R_z^{(02)}(-\varphi_0)R_y^{(02)}(\frac{\pi}{2} - \theta_1)R_z^{(02)}(\varphi_0). \end{aligned} \quad (10)$$

The Φ is a single qutrit phase gate which advances the phase of $|2\rangle$ without affecting $|0\rangle$ and $|1\rangle$ states in the qutrit. $\Gamma_2(\Phi) = \text{diag}\{I_8, e^{i\varphi}\}$ and its synthesis is shown in Fig. 13.

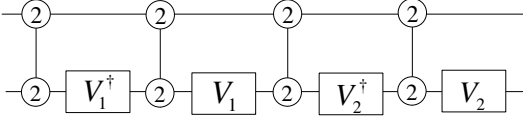


FIG. 12: Synthesis of Muthukrishnan-Stroud gate $\Gamma_2(\mathbb{Z})$.

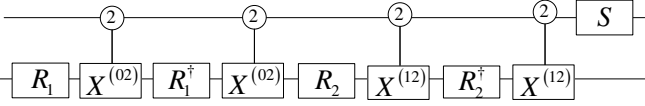


FIG. 13: Synthesis of Muthukrishnan-Stroud gate $\Gamma_2(\Phi)$, where $S = \text{diag}\{I_2, e^{i\varphi/3}\}$, $R_1 = R_z^{(02)}(\varphi/3)$, $R_2 = R_z^{(12)}(\varphi/3)$.

Based on the CSD, Khan and Perkowski investigate the structure of ternary quantum circuit. An arbitrary n qutrit gate can be synthesized with four multiplexers acting on $n - 1$ qutrits and three $(n - 1)$ -fold uniformly controlled rotations. The syntheses of these multiplexers and uniformly controlled rotations are much more complicated than that of Muthukrishnan-Stroud gate. We will investigate them in another article.

VI. GENERAL MULTI-VALUED CASE

In this section, we generalize the elementary gates to a general multi-valued quantum logic circuit case. We first extend the single qubit X gate to d -dimensional quantum systems (qudits) similar to the ternary case. The single qudit gate $X^{(jk)}$ is a gate which acts the X operation in two-dimensional subspace \mathcal{H}_{jk} of d -dimensional Hilbert

space. Similarly the definition of TCX gate can be naturally generalized to the qudit case. The generalized controlled X (GCX) gate is the two-qudit gate it implements the $X^{(ij)}$ operation on the target qudit iff the control qudit is in the states $|n\rangle$ ($n \in \{0, 1, \dots, d - 1\}$). Likewise, we can define the generalized controlled Z (GCZ) gate. The GCX gate or GCZ gate can be chosen as the two-qudit elementary gate, which is universal for qudit quantum computing, when it is assisted by arbitrary one-qudit gates. The GCX gate has d kinds of control mode and $\frac{1}{2}d(d - 1)$ different $X^{(ij)}$ operations. They can be transferred one another by the similar mode shown in Fig. 2. This holds true for the GCZ gate too.

The matrices of d -dimensional one-qudit gates are the elements of d -dimensional unitary group. From successive Cartan decompositions of $U(d)$ group and Euler decompositions, we can show that the set of one-qudit elementary gates has $d - 1$ pairs of basic gates acting on $d - 1$ different 2-dimensional subspaces. So manipulating a qudit completely needs at least $d - 1$ driving fields. The choice of $d - 1$ pairs of basic gates is not unique. They are universal if only the corresponding driving fields can connect the d levels of the qudit together. Like qutrit case, the pair of basic gates also has two modes: R_y , R_z and R_x , R_y modes. The Cartan decomposition of a one-qudit gate has given in appendix B.

VII. CONCLUSION AND FUTURE WORK

We have investigated the elementary gates for ternary quantum logic circuits. We propose the TCX or the TCZ gates as a two-qutrit elementary gate, with which arbitrary ternary quantum circuits can be synthesized when they are assisted by single qutrit gates. Based on the Cartan decomposition, the one-qutrit elementary gates are also investigated. They have two pairs of basic gates and two modes: $R_y^{(jk)}$, $R_z^{(jk)}$, $R_y^{(j'k')}$, $R_z^{(j'k')}$, or $R_x^{(jk)}$, $R_y^{(jk)}$, $R_x^{(j'k')}$, $R_y^{(j'k')}$. Then we have discussed the implementation scheme for the ternary elementary gates and have investigated the synthesis of some important ternary gates. The elementary gates proposed here are simple, efficient and can be implemented with current technology. Moreover, these elementary gates can be easily extended to a more general qudit case, so they constitute unified elements to synthesize quantum logic circuits, whatever they are qubit, qutrit, qudit or hybrid circuits. We can use them as a unified measure for the complexity of various quantum circuits.

The multi-valued and hybrid quantum computing is a new and exciting research area in which there is plenty of work to do. Moreover, to choose suitable quantum system, such as trapped ions, superconducting qutrits or qudits and quantum dots, to investigate the physical implementation of multi-valued quantum logic gates and to undertake the experimental work is crucial for the development of multi-valued quantum information science.

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Appendix A: CARTAN DECOMPOSITION OF ONE-QUTRIT GATES

The Cartan decomposition of a Lie group depends on the decomposition of its Lie algebras [25]. Let \mathfrak{g} be a semisimple Lie algebra and there is the decomposition relations

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}, \quad (\text{A1})$$

where \mathfrak{l} and \mathfrak{p} satisfy the commutation relations

$$[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}, [\mathfrak{l}, \mathfrak{p}] \subseteq \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{l}, \quad (\text{A2})$$

we said the decomposition is the Cartan decomposition of Lie algebra \mathfrak{g} . The \mathfrak{l} is closed under the Lie bracket, so it is a Lie subalgebra of \mathfrak{g} , and that $\mathfrak{p} = \mathfrak{l}^\perp$. A maximal Abelian subalgebra \mathfrak{a} contained in \mathfrak{p} is called a Cartan subalgebra. Then using the relation between Lie group and Lie algebra, every element M of Lie group G can be decomposed as

$$M = K_1 A K_2, \quad (\text{A3})$$

where $G = e^{\mathfrak{g}}$, $K_1, K_2 \in e^{\mathfrak{l}}$ and $A \in e^{\mathfrak{a}}$.

The one-qutrit gates form a 3-dimensional unitary group $U(3)$. We have 8 independent ternary Pauli's matrices: three $\sigma_x^{(ij)}$ matrices, three $\sigma_y^{(ij)}$ matrices, and the two independent $\sigma_z^{(ij)}$ matrices in the three of them. Multiplying these 8 independent Pauli's matrices by i , we get the basis vectors of Lie algebra $\mathfrak{su}(3)$ which we called the quasi-spin basis. Together with 3×3 identity matrix multiplied by i , they constitute the basis vectors of Lie algebra $\mathfrak{u}(3)$. Take a **AIII** type Cartan decomposition [25] of $\mathfrak{u}(3)$, that is

$$\mathfrak{u}(3) = \mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1))^\perp. \quad (\text{A4})$$

Lie subalgebra $\mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1))$ consists of subalgebra $\mathfrak{su}(2)$ and a complex basis $r = \text{diag}\{I_2, -2\} = 2\sigma_z^{(02)} - \sigma_z^{(01)}$. We choose

$$\mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1)) = \text{span}\{i(\sigma_x^{(01)}, \sigma_y^{(01)}, \sigma_z^{(01)}, r)\}, \quad (\text{A5})$$

$$\mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1))^\perp = \text{span}\{i(I_3, \sigma_x^{(02)}, \sigma_y^{(02)}, \sigma_x^{(12)}, \sigma_y^{(12)})\}, \quad (\text{A6})$$

and its Cartan subalgebra

$$\mathfrak{a} = \text{span}\{i(I_3, i\sigma_y^{(02)})\}. \quad (\text{A7})$$

So the one-qutrit matrix can be decomposed as

$$\begin{aligned} M &= e^{i\alpha} \tilde{M}_1^{(01)} R_z^{(01)} (-\theta) R_z^{(02)} (2\theta) R_y^{(02)} (\beta) \\ &\quad R_z^{(02)} (2\theta') R_z^{(01)} (-\theta') \tilde{M}_2^{(01)} \\ &= e^{i\alpha} M_1^{(01)} M^{(02)} M_2^{(01)}. \end{aligned} \quad (\text{A8})$$

Lie subalgebra and Cartan subalgebra of the Cartan decomposition can be different, so the decomposition is not unique, we can get more generic Eq. (5) in Sec.III.

Appendix B: CARTAN DECOMPOSITION OF ONE-QUIDIT GATES

The one-qudit gates form a $U(d)$ group. We can also use the quasi-spin basis. There are $\frac{1}{2}d(d-1)$ $\sigma_x^{(ij)}$ matrices, $\frac{1}{2}d(d-1)$ $\sigma_y^{(ij)}$ matrices and $d-1$ independent $\sigma_z^{(ij)}$ matrices in the $\frac{1}{2}d(d-1)$ of them for a n -dimensional Hilbert space. Multiplying these $d^2 - 1$ independent quasi-spin matrices by i , we gain the basis vectors of Lie algebra $\mathfrak{su}(d)$. Together with $d \times d$ identity matrix multiplied by i , they constitute the basis vectors of Lie algebra $\mathfrak{u}(d)$. We also take a kind of **AIII** type Cartan decomposition for $\mathfrak{u}(d)$, that is

$$\mathfrak{u}(d) = \mathfrak{s}(\mathfrak{u}(d-1) \oplus \mathfrak{u}(1)) + \mathfrak{s}(\mathfrak{u}(d-1) \oplus \mathfrak{u}(1))^\perp. \quad (\text{B1})$$

Lie algebra $\mathfrak{s}(\mathfrak{u}(d-1) \oplus \mathfrak{u}(1))$ consists of subalgebra $\mathfrak{su}(d-1)$ and a complex basis $r = \text{diag}\{I_{d-1}, -(d-1)\}$. We choose its Cartan subalgebra

$$\alpha = \text{span}\{i(I_d, \sigma_y^{(d-2, d-1)})\}. \quad (\text{B2})$$

So arbitrary one-qudit matrix can be expressed as

$$M = e^{i\alpha} K_1 R_y^{(d-2, d-1)} (\beta) K_2, \quad (\text{B3})$$

where $K_i \in S(U(d-1) \oplus U(1))$ group. The matrix M can be re-expressed as

$$\begin{aligned} M &= e^{i\alpha} \tilde{K}_1' e^{i\theta r} R_y^{(d-2, d-1)} (\beta) e^{i\theta' r} \tilde{K}_2' \\ &= e^{i\alpha} K_1' M^{(d-2, d-1)} K_2'. \end{aligned} \quad (\text{B4})$$

where $\tilde{K}_i', K_i' \in SU(d-1) \oplus 1$. That is, r can be expressed as a linear combination of $\sigma_z^{(jk)}$ s, $r = \sigma_z^{(0, d-2)} + \dots + \sigma_z^{(d-3, d-2)} + (d-1)\sigma_z^{(d-2, d-1)}$, so $e^{i\theta r}$ is a product of a series of $R_z^{(jk)}$ s. $R_y^{(d-2, d-1)}$ combines with $R_z^{(d-2, d-1)}$ in $e^{i\theta r}$ and $e^{i\theta' r}$ to form $M^{(d-2, d-1)}$, other $R_z^{(jk)}$ s are absorbed in K_i' s.

From Eq. (B4), we can see that the d -dimensional one-qudit elementary gates needs one pair of basic gate more than that for the $(d-1)$ -dimensional qudit. They come from Euler decompositions of $M^{(d-2, d-1)}$. The $(d-1)$ -dimensional qudit matrix K' can be decomposed further in same mode. The successive decomposition can be done until the qutrit occurs. The one-qutrit elementary gates are two pairs of basic gates, so we can infer that the set

of d -dimensional one-qudit elementary gates has $d - 1$ pairs of basic gates. Likewise the mode of Cartan decomposition and its Cartan subalgebra can be different,

the decomposition is not unique, so the choice of $d - 1$ pairs of basic gates is not unique too.

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